Axiomatizing Step-by-Step.

Lessons from Modal Information Logic

Søren Brinck Knudstorp
ILLC, University of Amsterdam
Selections from MSc thesis, supervised by Johan van Benthem and Nick Bezhanishvili

April 11, 2024

Tsing Ch'a Session at Tsinghua University

Plan for the talk

- **Big picture, few details** (so please let me know if you'd like elaboration)
- **Outline of the talk:**
  1. Introducing the logics
  2. Stating the problems
  3. Outlining the strategy
  4. Solving the problems using the strategy
- **Overarching themes:**
  1. General heuristics and specific methods for axiomatization (and the Finite Model Property (FMP) and decidability)
  2. A study of modal information logics

Defining (the basic) modal information logics (MILs)

Definition (language and semantics)

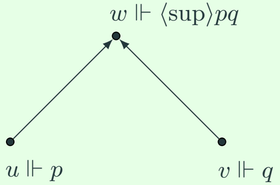
The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \sup \rangle \varphi\psi,$$

and the semantics of '$\langle \sup \rangle$' is:

$$w \Vdash \langle \sup \rangle \varphi\psi \quad \text{iff} \quad \exists u, v(u \Vdash \varphi;\ v \Vdash \psi;\ w = \sup\{u, v\})$$

Example



Definition (frames and logics)

Three classes of frames $(W, \leq)$, namely those where

(Pre) $(W, \leq)$ is a preorder (refl., tr.);

(Pos) $(W, \leq)$ is a poset (anti-sym. preorder); and

(Sem) $(W, \leq)$ is a join-semilattice (poset w. all bin. joins)

Resulting in the logics MIL_{Pre}, MIL_{Pos}, MIL_{Sem}, respectively.

Appetizer: Let's show that $MIL_{Pre} \subseteq MIL_{Pos} \subsetneq MIL_{Sem}$. **see blackboard**

Motivation

Why MILs?

- Connect with other logics (e.g., team and truthmaker semantics).
- Introduced to model a theory of information (by van Benthem (1996)).
- Modestly extend $\mathbf{S4}$ [MIL_{Pre}, MIL_{Pos}]. **see blackboard**

What in particular?

Guided by two central problems (posed in van Benthem (2017, 2019)), namely

(A) axiomatizing MIL_{Pre} and MIL_{Pos}; and

(D) proving (un)decidability.

Initial study (MIL_{Pre} and MIL_{Pos})

Proposition

MILs lack the finite model property (FMP) w.r.t. their classes of definition. **see blackboard**

How we solve (A), and then (D) using (A):

(1) We axiomatize MIL_{Pre} (and deduce $MIL_{Pre} = MIL_{Pos}$).

(2) Use the axiomatization to find another class of structures $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = MIL_{Pre}$.

(3) Prove that on $\mathcal{C}$ we do have the FMP and deduce decidability.

(1): axiomatizing *MIL*$_{Pre}$

Axiomatization (soundness and completeness)

MIL$_{Pre}$ is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

Proof idea

Soundness **see blackboard** ✓

For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent Γ_0. We construct a satisfying model using the step-by-step method (but first, why step-by-step? **see blackboard**).

(Base) Singleton frame $\mathbb{F}_0 := (\{x_0\}, \{(x_0, x_0)\})$ and 'labeling' $l_0(x_0) = \Gamma$.

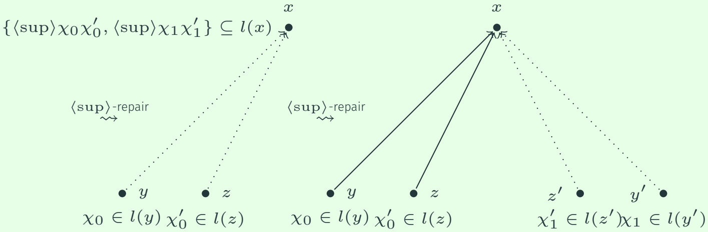
(Ind) Suppose $(\mathbb{F}_n, l_n)$ has been constructed.

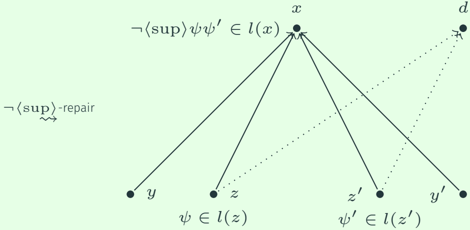
– If $x \in \mathbb{F}_n$ and $\neg\langle \text{sup} \rangle \psi\psi' \in l_n(x)$ but $x = \text{sup}_n\{y, z\}$ s.t. $\psi \in l_n(y), \psi' \in l_n(z)$, *coherently* extend to $(\mathbb{F}_{n+1}, l_{n+1}) \supseteq (\mathbb{F}_n, l_n)$ so that $x \neq \text{sup}_{n+1}\{y, z\}$.

– Similarly, for $\langle \text{sup} \rangle \chi\chi' \in l_n(x)$.

Completeness of MIL_{Pre} (cont.)

Example





(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

About the proof

Soundness: routine.

Completeness: step-by-step method.

Corollary

As a corollary we get that $MIL_{Pre} = MIL_{Pos}$.

(2) and (3): ‘decidability via completeness’

(2) **Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \textit{MIL}_{\textit{Pre}}$:**

(i) Nothing in the ax. of $\textit{MIL}_{\textit{Pre}}$ necessitating ‘$\langle \mathrm{sup} \rangle$’ to be interpreted using a supremum relation.

(ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \wedge (Co.) \wedge (4) \wedge (Dk.)\},$$

where $C \subseteq W^3$ is an arbitrary relation.

(iii) Then $\mathrm{Log}(\mathcal{C}) = \textit{MIL}_{\textit{Pre}}$. **see blackboard**

(3) **Decidability through FMP on $\mathcal{C}$:**

(i) On $\mathcal{C}$, we get the FMP through filtration.

(ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

Can we generalize these techniques?

MILs with informational implication '\'

(Natural) extensions of MIL_{Pre} and MIL_{Pos} [and $\mathbf{S4}$] are obtained by adding an informational implication '\'.

Definition

The language is given by adding '\' with semantics:

$$v \Vdash \varphi \backslash \psi \qquad \text{iff} \qquad \forall u, w([u \Vdash \varphi, w = \sup\{u, v\}] \Rightarrow w \Vdash \psi)$$

We denote the resulting logics as $MIL_{\backslash\text{-}Pre}$, $MIL_{\backslash\text{-}Pos}$, respectively.

Note that '\' is the residual of '$\langle \sup \rangle$'; and that 'F' is expressible: we extend temporal **S4**. **see blackboard**

The problems now become

(A\) axiomatizing $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$; and
(D\) proving (un)decidability.

The same (1)-(2)-(3) structure is used as before, but now we

(1') axiomatize the logic $\mathrm{Log}_{\backslash}(\mathcal{C})$;
(2') through representation show that $\mathrm{Log}_{\backslash}(\mathcal{C}) = MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$; and
(3) get decidability through FMP on $\mathcal{C}$.

Selected points from proof of (A\), (D\) through (1'), (2'), (3')

(1') axiomatizing $\mathrm{Log}_{\backslash}(\mathcal{C})$ (soundness and completeness)

$\mathrm{Log}_{\backslash}(\mathcal{C})$ is (sound and complete w.r.t.) the least set of $\mathcal{L}_{\backslash\text{-}M}$-formulas that (i) is closed under the axioms and rules for MIL_{Pre}; (ii) contains the K-axioms for \; (iii) contains the axioms

(I1) $\langle \sup \rangle p(p \backslash q) \to q$, and

(I2) $p \to q \backslash (\langle \sup \rangle pq)$;

and (iv) is closed under the rule

$(N_{\backslash})$ if $\vdash_{\backslash\text{-Pre}} \varphi$, then $\vdash_{\backslash\text{-Pre}} \psi \backslash \varphi$.

About the proof

Soundness: routine; completeness: standard.

Lambek Calculus of suprema on preorders/posets

$(MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos} =)$ $\mathrm{Log}_{\backslash}(\mathcal{C}) =$ **NL-CL** $+ \{(Re.), (4), (Co.), (Dk.)\}$,
where **NL-CL** is the Lambek Calculus extended with CL from, e.g., Buszkowski (2021).

MILs of minimal upper bounds

Question: What happens if we extend $\mathbf{S4}$ with vocabulary for *minimal* instead of *least* upper bounds?

Answer: Nothing. We get the exact same logics:

$$MIL_{Pre} = MIL_{Pos} = MIL^{Min}_{Pre} = MIL^{Min}_{Pos}$$

and even

$$MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos} = MIL^{Min}_{\backslash\text{-}Pre} = MIL^{Min}_{\backslash\text{-}Pos}$$

This concludes and summarizes our study of MILs on preorders and posets.

How about join-semilattices (i.e., MIL_{Sem})?
[Axiomatization problem raised by Bergman (2018) and Jipsen et al. (2021)]

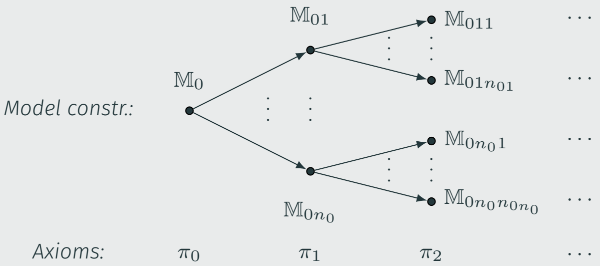
Axiomatizing MIL_{Sem}

Three ways to completeness (some intuitions for our proof):

Henkin (e.g., $\mathbf{K}$)

Standard step-by-step (e.g., MIL_{Pre})

'Indeterministic step-by-step' (MIL_{Sem})



Question: We've shown that $\mathit{MIL}_{\mathit{Sem}}$ is *infinitely* axiomatizable, but is it *finitely* axiomatizable?

Answer: No.

Conclusion

Summary:

- Surveyed the landscape of MILs on preorders and posets.[1]
- Made crossings with the Lambek calculus and truthmaker semantics.[2]
- Axiomatized MIL_{Sem} (and proved its non-finite axiomatizability).

Core messages:

- For semantically introduced logics:
    1. FMP absence may not be significant
    2. Decidability may be achievable 'via completeness'
- Axiomatizing step-by-step is flexible:
    1. Can be partial (adding dummies)
    2. Can be indeterministic (using König's Lemma)

[1] See SBK (2023b)

[2] See SBK (2023a) (or my thesis) for details, including FMP and decidability (and compactness) of a family of truthmaker logics.

References I

Bergman, C. (2018). "Introducing Boolean Semilattices". In: *Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science. Outstanding Contributions to Logic.* Ed. by J. Czelakowski. Springer, pp. 103–130 (cit. on p. 14).

Buszkowski, W. (03/2021). "Lambek Calculus with Classical Logic". In: *Natural Language Processing in Artificial Intelligence—NLPinAI 2020*, pp. 1–36. DOI: 10.1007/978-3-030-63787-3_1 (cit. on p. 12).

Jipsen, P., M. Eyad Kurd-Misto, and J. Wimberley (2021). "On the Representation of Boolean Magmas and Boolean Semilattices". In: *Hajnal Andréka and István Németi on Unity of Science: From Computing to Relativity Theory Through Algebraic Logic.* Ed. by J. Madarász and G. Székely. Cham: Springer International Publishing, pp. 289–312. DOI: 10.1007/978-3-030-64187-0_12 (cit. on p. 14).

References II

Knudstorp, S. B. (2023a). **"Logics of Truthmaker Semantics: Comparison, Compactness and Decidability".** In: *Synthese* (cit. on p. 17).

— (2023b). **"Modal Information Logics: Axiomatizations and Decidability".** In: *Journal of Philosophical Logic* (cit. on p. 17).

Van Benthem, J. (1996). **"Modal Logic as a Theory of Information".** In: *Logic and Reality. Essays on the Legacy of Arthur Prior.* Ed. by J. Copeland. Clarendon Press, Oxford, pp. 135–168 (cit. on p. 4).

— (10/2017). **"Constructive agents".** In: *Indagationes Mathematicae* 29. DOI: `10.1016/j.indag.2017.10.004` (cit. on p. 4).

References III

Van Benthem, J. (2019). "Implicit and Explicit Stances in Logic". In: *Journal of Philosophical Logic* 48.3, pp. 571–601. DOI: *10.1007/s10992-018-9485-y* (cit. on p. 4).

Thank you!

How can we think of this algebraically?

From relations to algebras

Given a preorder $(W, \leq)$, we can form its complex algebra w.r.t. the induced supremum relation:

$$(\mathcal{P}(W), \cap, \cup, ^c, \varnothing, W, \cdot),$$

where

$$Y \cdot Z := \{x \in W \mid x = \sup\{y, z\}, y \in Y, z \in Z\}.$$

Let Pre^+, Pos^+ and Sem^+ denote the classes of complex algebras of preorders, posets and (join-)semilattices w.r.t. the supremum relation. Then,

- $\mathit{MIL}_{\mathit{Pre}}$ corresponds to the variety $\mathbf{V}(\mathit{Pre}^+)$;
- $\mathit{MIL}_{\mathit{Pos}}$ to the variety $\mathbf{V}(\mathit{Pos}^+)$; and
- $\mathit{MIL}_{\mathit{Sem}}$ to the variety $\mathbf{V}(\mathit{Sem}^+)$.